Lattice gauge notes

Why quarks:
- eightfold way
- deep inelastic scattering
- charmonium

But free quarks not observed
- flux tube picture of confinement.
- not perturbative; turn to the lattice
- define a field theory
- allows computation

Asymptotic freedom review: two coupling definitions agree to lowest order,

$$g_1 = g_2 + Cg^3 + O(g^5) \tag{1}$$

Let the couplings be at two nearby momenta

$$q^2 \frac{dg}{dq^2} \equiv \beta(g) = -b_0 g^3 + b_1 g^5 + \dots$$

I put in the minus sign so b_0 is positive, i.e. in non-Abelian gauge theories.

$$\frac{dq^2}{q^2} = \frac{dg}{-b_0 g^3 + b_1 g^5 + \dots} = -\frac{dg}{b_0 g^3} - \frac{b_1 dg}{b_0 g} + \dots$$

Integrating

$$\log(q^2) = C + \frac{1}{2b_0 g^2} - \frac{b_1 \log(g)}{b_0} + \dots$$

Usually rewritten

$$g^2 = \frac{1}{2b_0 \log(q^2/\Lambda^2) + \dots}$$

so $g \to 0$ as $q^2 \to \infty$. But for a lattice guy, exponentiate to get

$$q^2 = \Lambda^2 e^{1/2b_0 g^2} g^{-b_1/b_0} (1 + O(g^2))$$

Work with bare coupling, so $q^2 \sim 1/a^2$. Thus the lattice spacing becomes a function of the input coupling

$$a^2 = \frac{1}{\Lambda_{lat}} e^{-1/2b_0 g^2} g^{b_1/b_0} (1 + O(g^2))$$

We adjust the lattice spacing via the bare coupling. The relation has a non-perturbative factor. Continuum limit $a \to 0$ corresponds to $g \to 0$.

At large coupling the meaning of the beta function becomes obscure. It depends on your definition of the coupling; different definitions can behave quite differently. The natural lattice definition of the beta function has a zero at large distances (details in lecture)!

- show b_0 and b_1 universal for any couplings related as in eq.(1)

Quantum mechanics in d dim is related to classical statistical mechanics in d+1 dim. Start with a single particle in a potential, i.e. zero dimensional quantum field theory.

$$H = \hat{p}^2/2 + V(\hat{x})$$

with $[\hat{p}, \hat{x}] = i$. Look at the quantum partition function

$$Z = \mathrm{Tr} e^{-\beta H}$$

- find Z for the harmonic oscillator with $V = X^2/2$

Divide "imaginary time" β into slices:

$$Z = \mathrm{Tr} \left(e^{-\beta H/N}\right)^N$$

Now insert a complete set of states, $1 = \int dx|x\rangle\langle x|$ between each factor

$$Z = \int dx_0 \ldots dx_{N-1} \prod_i \langle x_{i+1}|e^{-\beta H/N}|x_i\rangle$$

Here to get the trace $x_N = x_0$, i.e. we have periodic boundary conditions.

For N large, approximate

$$e^{-\beta H/N} \sim e^{-\beta V(\hat{x})/2N} e^{\beta \hat{p}^2/2N} e^{-\beta V(\hat{x})/2N}$$

so that

$$\langle x'|e^{-\beta H/N}|x\rangle \sim e^{-\beta(V(x)+V(x'))/2N} \langle x'|e^{\beta \hat{p}^2/2N}|x\rangle$$

For the last factor, use $1 = \int dp|p\rangle\langle p|$ and $\langle p|x\rangle = e^{ipx}$

$$\langle x'|e^{\hat{p}^2/2N}|x\rangle = \int dp e^{p^2/2} e^{ip(x-x')} = \sqrt{2\pi/\beta} e^{-(x-x')^2 N/2\beta}$$

Putting it all together

$$Z = A \int dx_0 \ldots dx_{N-1} e^{-\beta S}$$

with A an irrelevant factor and

$$S = \frac{1}{N} \sum_t (x_{t+1} - x_t)^2 N^2/2\beta^2 + V(x_t)$$

Note the analogies:

$$\beta/N \leftrightarrow dt$$
$$\beta/N \sum_t \leftrightarrow \int dt$$
$$(x_{t+1} - x_t)N/\beta = \frac{(x_{t+1} - x_t)}{\beta/N} \leftrightarrow \dot{x} = \frac{dx}{dt}$$
$$S \leftrightarrow \int dt \ \dot{x}^2/2 + V(x)$$

- this is a lattice theory $a \sim 1/N$
- derivatives become nearest neighbor differences
- QM equivalent to classical statistical mechanics in one more dimension
- imaginary time natural, e^{iH} versus $e^{-}\beta H$
- the same H; $\beta \to \infty$ gives ground state
- $\langle (x' - x)^2/a^2 \rangle$ diverges

Transfer matrix notation:

- $T_{x',x} = \exp((x' - x)^2 N/2\beta + \beta(V(x') + V(x))/2)$
- $Z = \mathrm{Tr} T^N$
- $Ha_t \leftrightarrow \log(T)$
- relates Hamiltonian and Lagrangian formulations

Study the free field theory with "continuum" action

$$S = \int d^4x \ (\partial_\mu \phi)^2 + m^2 \phi^2/2$$

Put on a lattice of discrete points

$$x_\mu = a n_\mu$$

with n having only integer components. Let the lattice have length L in each dimension, so the physical volume is $a^4 L^4$. Use periodic boundaries. All fields are encountered when $0 <= n_\mu < L$.

A natural transcription is

$$\partial_\mu \phi \longrightarrow \frac{\phi_{n+e_\mu} - \phi_n}{a}$$

To keep notation simple, let $\{m, n\}$ denote the set of nearest neighbor sites, each pair taken once. The the action is

$$\begin{aligned} S &= a^2 \sum_{\{m,n\}} \frac{(\phi_m - \phi_n)^2}{2} + a^4 m^2 \sum_n \frac{\phi^2}{2} \\ &= a^2 \sum_{\{m,n\}} -\phi_m \phi_n - \phi_n \phi_m + a^2(2 + a^2 m^2) \sum_n \frac{\phi_n^2}{2} \end{aligned}$$

Note, redefining $\phi_l = a\phi$ and $m_l = am$ would remove all factors of the lattice spacing. These would be natural "lattice units."

Now for the path integral:

$$Z = \int \prod_n d\phi_n e^{-S}$$

Formally, S is a quadratic form,

$$S = \frac{1}{2} \phi M \phi = \frac{1}{2} \sum_{mn} \phi_m M_{mn} \phi_n$$

where the Hermitian matrix M is

$$M_{mn} = -a^2 \sum_\mu (\delta_{m,n+e_\mu} + \delta_{m,n-e_\mu}) + a^2(2 + a^2 m^2)\delta_{m,n}$$

and we can write

$$Z = |M/2\pi|^{1/2}$$

Fourier transforms based on summing roots of unity

$$\sum_{n=0}^{L-1} e^{2\pi i n k/L} = L\delta_{k,0}$$

for $k \in \{0, ...L-1\}$. Thus motivated, define

$$\tilde{\phi}_k = \sum_n e^{2\pi i n\cdot k/L}\phi_n$$

Inversion is simply

$$\phi_n = \frac{1}{L^4}\sum_k e^{-2\pi i n\cdot k/L}\tilde{\phi}_k$$

The measures are related by a constant factor

$$\int (d\phi) = \int \prod_n d\phi_n = (L^2)^{-V}\int (d\tilde{\phi})$$

(Note: for our real field, $\tilde{\phi}_k = \tilde{\phi}^*_{-k}$.)

This makes some sums diagonal:

$$\sum_n \phi^*_n\phi_{n+m} = \frac{1}{L^4}\sum_k e^{-2\pi i m\cdot k/L}\phi^*_k\phi_k$$

and the action takes a really simple form

$$S = \frac{a^2}{L^4}\sum_k (2 + a^2m^2 - 2\cos(2\pi k/L))\phi^*_k\phi_k/2$$

The partition function is

$$Z = \prod_k \left(\frac{L^2(2 + a^2m^2 - 2\cos(2\pi k/L))}{2\pi a^2}\right)^{1/2}$$

For propagators:

$$\langle\phi(x)\phi(y)\rangle = \int (d\phi)\phi(x)\phi(y)e^{-S}/Z$$

goes over to

$$\langle\phi_n\phi_m\rangle = \frac{1}{L^4a^2}\sum_k e^{-2\pi i k\cdot(n-m)/L}\frac{1}{2 + a^2m^2 - 2\cos(2\pi k/L)}$$

- Homework: verify this

Connection to the continuum: $2\pi k \cdot n/L \leftrightarrow q \cdot x$ so with $x = an$ we identify $q = 2\pi k/aL$

- finite volume makes momentum discrete, steps $2\pi/aL$
- shift k by $L/2$ for symmetry, i.e. $-L/2 < k_\mu \leq L/2$
- lattice gives a cutoff $|q| < \pi/a$
- $\frac{d^4 q}{(2\pi)^4} \leftrightarrow \frac{1}{a^4 L^4} \sum_k$
- $q^2 + m^2 \leftrightarrow (2 - 2\cos(qa) + a^2 m^2)/a^2 = q^2 + m^2 + O(q^4 a^2)$
- lattice artifacts higher order in the spacing
- all factors of momenta get replaced by trig functions

Digression on Fourier transforms (revert to 1 dim):

$$\tilde{f}_k = \sum_{n=0}^{L-1} e^{2\pi i k n/L} f_n$$

Given L values of k, there are L terms in the sum, so the work to calculate this appears to be $O(L^2)$. Suppose L is even and rewrite this as a sum of the even terms plus the odd terms

$$\begin{aligned} \tilde{f}_k &= \sum_{j=0}^{L/2-1} e^{2\pi i k(2j)/L} f_{2j} + \sum_{j=0}^{L/2-1} e^{2\pi i k(2j+1)/L} f_{2j+1} \\ &= \sum_{j=0}^{L/2-1} e^{2\pi i k j/(L/2)} f_{2j} + e^{2\pi i k/L} \sum_{j=0}^{L/2-1} e^{2\pi i k j/(L/2)} f_{2j+1} \end{aligned}$$

Each term is now a fourier transform on a system of size $L/2$. Once these are done, then they need be added together for each k, i.e. work going as L. But for each factor of two in L this can be repeated. If L is a power of two, we can go all the way to a fourier transform on a single site lattice; that is just a copy. So the total work is

$$L + 2(\frac{L}{2} + 2(\frac{L}{4} + ...)) = L + 2\frac{L}{2} + 4\frac{L}{4} + 8\frac{L}{8} \ldots$$

where there are $\log_2(L)$ terms in this sum. Thus the real work needed is $L \log_2(L)$, which can be much less than the naive L^2. This recursive procedure is the famous fast fourier transform algorithm.